Proofs of a Trigonometric Inequality

Abstract

A trigonometric inequality is introduced and proved using Hölder's inequality, Cauchy-Schwarz inequality, and Chebyshev's order inequality.

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1. Introduction

Trigonometric inequalities are very important in many mathematical areas. Because of its wide and profound application, it has become a popular research interest.

Lohwarter mentioned in his book the following two inequalities (see [2], p.5 and p.78):

$$\sin^4\theta + \cos^4\theta \geq \frac{1}{2}$$

and

$$\sin^6\theta + \cos^6\theta \geq \frac{1}{4}$$

when $0 \leq \theta \leq \frac{\pi}{2}$. Naturally, one may ask if the above results can be generalized. By studying their proofs, we found a pattern and successfully derived a more generalized form:

$$\sin^{2(n+1)}x + \cos^{2(n+1)}x \geq \frac{1}{2^n}, \tag{1}$$

where x is a real number and n is a non-negative integer. In this paper, applying different known inequalities, we will provide several proofs of this new inequality.

We use Hölder's inequality in our first proof. For the case when $x \in [\,0,\ \frac{\pi}{2}\,]$, because of the non-negativity of $\sin x$ and $\cos x$, we also derive another similar trigonometric inequality:

$$\sin^{n+1}x + \cos^{n+1}x \geq \frac{1}{2^{\left(\frac{n-1}{2}\right)}},$$

where n is a positive integer. As a special case of Hölder's inequality, Cauchy-Schwarz inequality is used in our second proof. And in the third proof, we apply Chebyshev's order inequality. Not being able to complete the whole proof, however, we apply Jensen's inequality and prove a special case of (1) when $n = 2^i - 1$, where i is a non-negative integer.

2. Proofs

In this section we will provide three proofs of the following main result.

Theorem 1. *For any real number x and a non-negative integer n, we have the inequality*

$$\sin^{2(n+1)}x + \cos^{2(n+1)}x \geq \frac{1}{2^n}.$$

We notice that when $n = 0$, the above inequality is equivalent to the well-known Pythagorean identity $\sin^2 x + \cos^2 x = 1$. Therefore in our three proofs we omit this trivial case, and will only prove the inequality when $n \geq 1$. Also, the requirement for the equality is the same for all three proofs, hence will only be discussed in the first proof.

Proof 1. We start with Hölder's inequality. The Hölder's inequality states, if p and q are two real numbers in the interval $(1, \infty)$ with $\frac{1}{p} + \frac{1}{q} = 1$, we have

$$\sum_{i=1}^{i=k} |a_i b_i| \leq \left(\sum_{i=1}^{i=k} |a_i|^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{i=k} |b_i|^q\right)^{\frac{1}{q}}$$

for any two real number sequences $\{a_1, a_2, a_3, \cdots, a_k\}$ and $\{b_1, b_2, b_3, \cdots, b_k\}$. And the equality holds if $|a_i|^p = c|b_i|^q$, where c is a real number.

We first notice that for non-negative real numbers a and b with $a + b = 1$, and a positive integer n, according to Hölder's inequality we have

$$(a^{n+1} + b^{n+1})^{\frac{1}{n+1}} \left(1^{\frac{n+1}{n}} + 1^{\frac{n+1}{n}}\right)^{\frac{n}{n+1}} \geq a + b,$$

which is equivalent to

$$a^{n+1}+b^{n+1}\geq \frac{1}{2^n}.$$

Since $\sin^2 x$ and $\cos^2 x$ are both non-negative and $\sin^2 x+\cos^2 x=1$, substituting a and b with $\sin^2 x$ and $\cos^2 x$ respectively we then have

$$\sin^{2(n+1)}x+\cos^{2(n+1)}x\geq \frac{1}{2^n}.$$

In the proof, the equality occurs when $\sin^2 x=\cos^2 x=\frac{1}{2}$, or equivalently $x=\frac{\pi}{4}+\frac{k\pi}{2}$ for any integer k. ■

In proof 1, we take advantage of the non-negativity of both $\sin^2 x$ and $\cos^2 x$ for any real number x. If we use the restricted x that $0\leq x\leq \frac{\pi}{2}$, both $\sin x$ and $\cos x$ are non-negative, hence $\sin^{n+1}x=(\sin^2)^{\frac{n+1}{2}}$ and $\cos^{n+1}x=(\cos^2 x)^{\frac{n+1}{2}}$ for any n, we then can use the same technique to derive the following result.

Theorem 2. *For any real number* $x\in[0,\frac{\pi}{2}]$, *and a positive integer* n, *we have the inequality*

$$\sin^{n+1}x+\cos^{n+1}x\geq \frac{1}{2^{\left(\frac{n-1}{2}\right)}}.$$

Proof of Theorem 2. For any integer $n>1$, still applying Hölder's inequality we have

$$\left((\sin^2 x)^{\frac{n+1}{2}}+(\cos^2 x)^{\frac{n+1}{2}}\right)^{\frac{2}{n+1}}\cdot\left(1^{\frac{n+1}{n-1}}+1^{\frac{n+1}{n-1}}\right)^{\frac{n-1}{n+1}}\geq(\sin^2 x+\cos^2 x).$$

After simplification we get

$$\sin^{n+1}x+\cos^{n+1}x\geq \frac{1}{2^{\left(\frac{n-1}{2}\right)}}.$$

Together with the trivial case when $n=1$, $\sin^2 x+\cos^2 x=1$, that completes the proof of theorem 2. In this case, the equality holds only when $x=\frac{\pi}{4}$. ■

Remark. Unlike theorem 1, n has to be an integer greater than or equal to 1 in theorem 2. If $n=0$, $\sin x+\cos x=\sqrt{2}\sin\left(x+\frac{\pi}{4}\right)$, which values from 1 to $\sqrt{2}$ if $x\in[0,\frac{\pi}{2}]$. The inequality fails.

Proof 2. In our second proof of theorem 1, we will use strong induction and the Cauchy-Schwarz inequality.

Recall that the Cauchy-Schwarz inequality states, for two sets of real numbers $\{a_1, a_2, a_3, \cdots, a_k\}$ and $\{b_1, b_2, b_3, \cdots, b_k\}$, we have

$$\left(\sum_{i=1}^{i=k} {a_i}^2\right)\left(\sum_{i=1}^{i=k} {b_i}^2\right) \geq \left(\sum_{i=1}^{i=k} a_i b_i\right)^2,$$

where the equality holds if $a_i = c \cdot b_i$ for all index i and a real number c.

If $n = 1$, using Cauchy-Schwarz inequality we have

$$(1 + 1)(\sin^4 x + \cos^4 x) \geq (\sin^2 x + \cos^2 x)^2.$$

Equivalently, $\sin^4 x + \cos^4 x \geq \frac{1}{2}$. The inequality is true.

If $n = 2$, we notice that

$$\begin{aligned}\sin^6 x + \cos^6 x &= (\sin^2 x + \cos^2 x)(\sin^6 x + \cos^6 x) \\ &\geq (\sin^4 x + \cos^4 x)^2 \\ &\geq \frac{1}{2^2}.\end{aligned}$$

The inequality is still true.

Assume that the inequality is true for $n = 1, 2, \cdots, k$. If $k = 2i + 1$, Cauchy-Schwarz inequality provides us the following.

$$\begin{aligned}(1^2 + 1^2)\left(\sin^{2(k+1)} x + \cos^{2(k+1)} x\right) &\geq (\sin^{k+1} x + \cos^{k+1} x)^2 \\ &= \left(\sin^{2(i+1)} x + \cos^{2(i+1)} x\right)^2 \\ &\geq \left(\frac{1}{2^i}\right)^2\end{aligned}$$

for $i + 1 < 2i + 1 = k$. As a result,

$$\sin^{2(k+1)} x + \cos^{2(k+1)} x \geq \frac{1}{2^{2i+1}} = \frac{1}{2^k}.$$

If $k = 2i$, using Cauchy-Schwarz inequality again we have

$$\begin{aligned}\sin^{2(k+1)} x + \cos^{2(k+1)} x &= (\sin^2 x + \cos^2 x)\left(\sin^{2(k+1)} x + \cos^{2(k+1)} x\right) \\ &\geq (\sin^{k+2} x + \cos^{k+2} x)^2 \\ &= \left(\sin^{2(i+1)} x + \cos^{2(i+1)} x\right)^2.\end{aligned}$$

Because $i + 1 \leq 2i = k$, according to our assumption, $\sin^{2(i+1)} x + \cos^{2(i+1)} x \geq \frac{1}{2^i}$.

Therefore

$$\sin^{2(k+1)}x + \cos^{2(k+1)}x \geq \frac{1}{2^{2i}} = \frac{1}{2^k}.$$

Based on strong induction, the inequality is true for all integers $n \geq 1$. ■

Proof 3. We will use induction and Chebyshev's order inequality to complete the third proof.

The Chebyshev's order inequality states, for any two real number sequences $a_1 \leq a_2 \leq \cdots \leq a_k$ and $b_1 \leq b_2 \leq \cdots \leq b_k$,

$$\frac{1}{k}\sum_{i=1}^{i=k} a_i b_i \geq \left(\frac{1}{k}\sum_{i=1}^{i=k} a_i\right)\left(\frac{1}{k}\sum_{i=1}^{i=k} b_i\right).$$

And the equality holds if $a_i = a_j$ or $b_i = b_j$ for any $i \neq j$.

We first want to prove the case when $n = 1$. Applying Chebyshev's order inequality on the ordered pair $\{\sin^2 x, \cos^2 x\}$ we have

$$\frac{\sin^4 x + \cos^4 x}{2} \geq \frac{\sin^2 x + \cos^2 x}{2} \cdot \frac{\sin^2 x + \cos^2 x}{2},$$

which can be simplified to

$$\sin^4 x + \cos^4 x \geq \frac{1}{2}.$$

The claimed inequality is true.

Assume that the inequality is true when $n = k$. Because $\{\sin^2 x, \cos^2 x\}$ and $\{\sin^{2k} x, \cos^{2k} x\}$ are both increasing or decreasing, using Chebyshev's order inequality again we then have

$$\frac{\sin^{2k+2} x + \cos^{2k+2}}{2} \geq \frac{\sin^2 x + \cos^2 x}{2} \cdot \frac{\sin^{2k} x + \cos^{2k} x}{2}.$$

Equivalently,

$$\sin^{2(k+1)}x + \cos^{2(k+1)} \geq \frac{1}{2} \cdot \frac{1}{2^{k-1}} = \frac{1}{2^k}.$$

According to induction, the claimed inequality is then proved. ■

We now consider a special case. For any real number x and any integer $n = 2^i - 1$, where i is a non-negative integer, we can use Jensen's inequality to prove inequality (1).

Proof 4, Special Case of Theorem 1 when $n = 2^i - 1$. Recall that the Jensen's inequality states, if $f(x)$ is a convex function, for any positive real numbers α and β with $\alpha + \beta = 1$, we have

$$\alpha f(x_1) + \beta f(x_2) \geq f(\alpha x_1 + \beta x_2),$$

where x_1 and x_2 are two real numbers in the domain of $f(x)$. The equality holds if $x_1 = x_2$.

First, we re-write inequality (1) as follow:

$$\sin^{2^{(i+1)}} x + \cos^{2^{(i+1)}} x \geq \frac{1}{2^{(2^i - 1)}}.$$

(2)

If $i = 0$, inequality (2) is apparently true.

Assume that (2) is true for $i = k$. Consider function $f(x) = x^2$. This function is convex on interval $[0, 1]$. For any real number x, there exist x_1 and x_2 in $[0,1]$ such that $x_1 = \sin^{2^{(k+1)}} x$ and $x_2 = \cos^{2^{(k+1)}} x$. Applying Jensen's inequality,

$$\frac{f(x_1) + f(x_2)}{2} \geq f\left(\frac{x_1 + x_2}{2}\right),$$

that is

$$\begin{aligned} \frac{\sin^{2^{(k+2)}} x + \cos^{2^{(k+2)}} x}{2} &\geq \left(\frac{\sin^{2^{(k+1)}} x + \cos^{2^{(k+1)}} x}{2}\right)^2 \\ &\geq \left(\frac{1}{2^{(2^k)}}\right)^2 \\ &= \frac{1}{2^{\left(2^{(k+1)}\right)}}, \end{aligned}$$

which is equivalent to

$$\sin^{2^{(k+2)}} x + \cos^{2^{(k+2)}} x \geq \frac{1}{2^{(2^{(k+1)}) - 1}}.$$

According to induction again, inequality (2) is then proved. ■

References

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